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Regularized Integration of Gravity-Perturbed Trajectories—A Numerical Efficiency Study

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Abstract

NINE methods for predicting the motion of a particle in a perturbed gravity field, many of which are based upon regularizing transformations and some of which are newly developed, are numerically compared. The class of motions studied in detail is the fractional-orbit case typical of ballistic trajectories. Results from 12 test trajectories in an oblate gravity field indicate that an asymptotic expansion technique based upon an Euler parameter regularization is extremely efficient when compared to standard Cowell and Encke methods. Additionally, algorithms based upon the Kustaanheimo-Stiefel (KS) transformation are found to be numerically attractive.

Contents

Since the analytical development of the equations of motion associated with most of the methods is very lengthy, this synoptic will briefly summarize the techniques with appropriate comments and references and then report the test results. For a full developmental treatment, please refer to Ref. 1.

Method 1: Cowell Integration of the Equations of Motion in Rectangular Variables

The well-known equations of orbital or suborbital motion in rectangular variables (x,y,z) are

$$\ddot{x}+\mu\frac{x}{r^3}=-\frac{\partial V}{\partial x}+P_x^*=P_x, \qquad x\to y,z \tag{1}$$

where

(P_x^*,P_y^*,P_z^*) = perturbing accelerations not derivable from a potential energy function V
(P_x,P_y,P_z) = total perturbing accelerations

and

$$(\dot{\ }) = \frac{d}{dt}(\ ) \tag{2}$$

Method 2: Integration of Encke-Type Equations Associated with Rectangular Variables

The vector form of the equations of motion in rectangular variables is

$$\ddot{r}+\mu(r/r^3)=P \tag{3}$$

In the absence of non two-body perturbations, Eq. (3) acquires the form

$$\ddot{r}_n+\mu(r_n/r_n^3)=0 \tag{4}$$

which has the f and g solution form[2]

$$r_n=f_n(t)r_0+g_n(t)\dot{r}_0 \tag{5}$$

In the usual Encke style, the reference motion is chosen to be Keplerian so that the perturbed motion can be written as

$$r=r_n+\delta \tag{6}$$

The standard development leads to

$$\ddot{\delta}=(\mu/r_n^3)[hr_n+\delta(h-1)]+P \tag{7}$$

(See Refs. 3 and 4 for details and definitions of variables.)

At this point, the transformation of Eq. (7) from the independent variable t to that of $\Phi=E-E_0$ (change in eccentric anomaly) is investigated. Differentiation of Kepler's equation results in the relationship

$$dt=r\sqrt{a/\mu}dE=r\sqrt{a/\mu}d\Phi \tag{8}$$

Using Φ in lieu of t as the time-like variable typically allows the first-order scalar system equivalent to Eq. (7) to be integrated more efficiently.

Method 3: An Encke/Variation of Parameters Approach in Rectangular Variables

Rather than a straightforward time integration of the Encke equations (7) or the more efficient integration via the independent variable change [Eq. (8)], consider an idea presented by Broucke[5] and assume that the solution of Eq. (7) is

$$\delta=f_n(t)U(t)+g_n(t)V(t) \tag{9}$$

Performing a standard variation of parameters treatment with the assumed solution form [Eq. (9)], one arrives at

$$\dot{U}=-g_n\left[\frac{\mu}{r_n^3}(f_nU+g_nV+hr_n)+P\right] \tag{10}$$

and

$$\dot{V}=f_n\left[\frac{\mu}{r_n^3}(f_nU+g_nV+hr_n)+P\right] \tag{11}$$

Equations (10) and (11) could themselves be numerically integrated to determine instantaneously valid $U(t)$ and $V(t)$ for use in Eq. (9); once δ is known, Eq. (6) furnishes the desired position. However, using the independent variable change [Eq. (8)] of the last section, Eqs. (10) and (11) become

$$\mathring{U}=-r_n\sqrt{\frac{a}{\mu}}g_n\left[\frac{\mu}{r_n^3}(f_nU+g_nV+hr_n)+P\right] \tag{12}$$

and

$$\mathring{V}=r_n\sqrt{\frac{a}{\mu}}f_n\left[\frac{\mu}{r_n^3}(f_nU+g_nV+hr_n)+P\right] \tag{13}$$

where

$$(\mathring{\ })=(d/d\Phi)(\ ) \tag{14}$$

Observe that using U and V as elements is attractive since they are both small and slowly varying.

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EDITOR'S NOTE:
This synoptic is in excess of the standard length owing to it's timeliness and mathematical completeness.

Method 4: Cowell Integration of the Equations of Motion in Kustaanheimo-Stiefel (KS) Variables

The dependent variable transformation

$$\begin{Bmatrix} x \\ y \\ z \\ 0 \end{Bmatrix} = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{15}$$

written compactly as

$$r = L(u)u \tag{16}$$

and the time variable transformation

$$dt = r ds \tag{17}$$

bring Eqs. (1) to the form[6]

$$\overset{\circ\circ}{u} + \alpha_T u = Q \tag{18}$$

$$\overset{\circ}{\alpha}_T = -\left(\frac{du}{ds}\right)^T L^T(u) P^* - \frac{r}{2}\left(\frac{\partial V}{\partial t}\right) \tag{19}$$

where

$$\alpha_T = \frac{1}{r}\left(\frac{\mu}{2} - \overset{\circ}{u}^T \overset{\circ}{u}\right) - \frac{V}{2} \tag{20}$$

$$Q = -\frac{1}{4}\frac{\partial}{\partial u}(rV) + \frac{r}{2} L^T(u) P \tag{21}$$

$$r = u^T u \tag{22}$$

and

$$(\,^\circ\,) = \frac{d}{ds}(\ ) \tag{23}$$

Note that if $P=0$, then α_T [$=-2$ (total energy)] is constant; if V is also zero (Keplerian motion), the result is four uncoupled linear oscillators. In the presence of small perturbations, the oscillators become weakly nonlinear with weak coupling and a slowly varying frequency.

Method 5: An Encke Approach in KS Variables

Utilizing a straightforward Encke scheme, we let

$$u = u_R + \delta u \tag{24}$$

$$t = t_R + \delta t \tag{25}$$

where the subscript R denotes a reference Keplerian quantity and δ the departure quantity. Substitution of Eq. (24) into Eq. (18) yields

$$\overset{\circ\circ}{u}_R + \delta\overset{\circ\circ}{u} + \alpha_T(u_R + \delta u) = -\tfrac{1}{2} V u + (r/2) L^T(u) P \tag{26}$$

We note that for conservative perturbations α_T is constant along the perturbed path and can be evaluated at the initial or osculation time by

$$\alpha_T = \alpha_k + V_0/2 \tag{27}$$

where α_k is the energy associated with the reference Keplerian path and V_0 the perturbed potential evaluated initially. Substituting Eq. (27) into Eq. (26) and noting that the reference motion satisfies

$$\overset{\circ\circ}{u}_R + \alpha_k u_R = 0 \tag{28}$$

one arrives at

$$\delta\overset{\circ\circ}{u} + \alpha_k \delta u = \tfrac{1}{2}(V_0 - V)u + (r/2) L^T(u) P \tag{29}$$

Substitution of Eq. (25) into Eq. (17) gives

$$\delta\overset{\circ}{t} = r - r_R \tag{30}$$

Method 6: Cowell Integration of the Equations of Motion in Burdet Variables

The transformations

$$\hat{x} = r/r = \rho r \tag{31}$$

and

$$dt = (1/\sqrt{\mu p^*}) r^2 d\phi \tag{32}$$

bring Eqs. (3) to the regularized "Burdet oscillator" differential equations of tenth order:

$$\frac{d^2\hat{x}}{d\phi^2} + \hat{x} = \frac{1}{\mu p^* \rho^3}[P - (P\cdot\hat{x})\hat{x}] - \frac{1}{2p^*}\frac{dp^*}{d\phi}\frac{d\hat{x}}{d\phi} \tag{33}$$

$$\frac{d^2\rho}{d\phi^2} + \rho = \frac{1}{p^*} - \frac{1}{\mu p^* \rho^2}(P\cdot\hat{x}) - \frac{1}{2p^*}\left(\frac{dp^*}{d\phi}\right)\frac{d\rho}{d\phi} \tag{34}$$

$$\frac{dp^*}{d\phi} = \frac{2}{\mu\rho^3}\left(P\cdot\frac{d\hat{x}}{d\phi}\right) \tag{35}$$

$$\frac{dt}{d\phi} = \frac{1}{\sqrt{\mu p^*}}\frac{1}{\rho^2} \tag{36}$$

In the above equations, p^* is the semilatus rectum of the osculating orbit and ϕ the change in true anomaly.

Method 7: An Encke Approach in Burdet Variables

In the absence of perturbations, Eqs. (33-35) are linear and Vitins[7] has provided a solution of Eq. (36). As was the case with KS variables, it is natural to adopt an Encke approach with the reference variables being associated with Keplerian motion. The procedure is straightforward; see Ref. 1 for full details.

Method 8: Cowell Integration of the Equations of Motion in Euler Parameter Variables

If the orbit normal $\hat{z}$, the radial unit vector $\hat{x}$, and $\hat{y} = \hat{z}\times\hat{x}$ are considered to be a rotating triad, it is possible to introduce rotational kinematics to the orbital problem. One may relate $\hat{x}$, $\hat{y}$, and $\hat{z}$ to the inertial unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ via

$$\begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{Bmatrix} = [C] \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \tag{37}$$

where the transformation matrix $[C]$ may be expressed as a function of the Euler parameters

$$U_i = l_i \sin(\psi/2)_2 \qquad i = 1,2,3 \qquad U_4 = \cos(\psi/2) \tag{38}$$

l_i being the direction cosines of the principal line of rotation and ψ the principal line of Euler's principal rotation theorem. Vitins[7] (see also Ref. 8) has pursued the above description of mass position along with the variable changes

$$\hat{x} = r/r \tag{39}$$

$$\rho = 1/r \tag{40}$$

$$p = p^* + (2r^2/\mu) V \tag{41}$$

$$dt = \frac{r^2 \nu}{\sqrt{\mu p^*}} d\phi = \frac{1}{\sqrt{\mu p}}\frac{1}{\rho^2} d\phi \tag{42}$$

$$\nu = \sqrt{\frac{p^*}{p}} = \sqrt{1 - \frac{2r^2}{\mu p} V} \tag{43}$$

The resulting eighth order system is

$$\begin{Bmatrix} \overset{\circ}{U}_1 \\ \overset{\circ}{U}_2 \\ \overset{\circ}{U}_3 \\ \overset{\circ}{U}_4 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} \tag{44}$$

$$\overset{\circ\circ}{\rho} + \rho = (1/p) + g_1 \tag{45}$$

$$\overset{\circ}{p} = g_2 \tag{46}$$

$$\overset{\circ}{t} = (1/\sqrt{\mu p})(1/\rho^2) \tag{47}$$

where $(^\circ) = d/d\phi$, the ω_i are angular velocity components of $(\hat{x}, \hat{y}, \hat{z})$ relative to inertial space, and g_1 and g_2 are functions of P^* and V which vanish for Keplerian motion.

Method 9: An Asymptotic Expansion Technique Associated with the Equations of Motion in Euler Parameter Variables

Since Eqs. (44-46) have an analytical Keplerian solution [Eq. (47) reduces to the classical Kepler equation], a perturbation solution may result in computational efficiencies. Each variable in Eqs. (44-46) may be written as a power series expansion with J_2 as the small parameter. In Ref. 1, it is shown that closed-form finite Fourier series solutions for the first-order quantities result from this approach. Strictly speaking, of course, method 9 belongs to a different class of methods than the first eight, since method 9 involves analytical integration whereas the first eight all involve numerical integration of competing differential equations. However, method 9 is included because it is new and interesting and it provides a normalizing comparison for analytical vs numerical solutions of the same test cases.

Test Results

Methods 1-9 were applied to 12 test ballistic J_2 trajectories. The selected accuracy criterion is the ability of each integration procedure to produce a reference final state, which was determined by reducing the step size used in method 1 until five decimal places ($\sim 10^{-2}$ m) stabilized in $x(t_f)$, $y(t_f)$, and $z(t_f)$. With the reference final state $[x_R(t_f), y_R(t_f), z_R(t_f)]$ established, the step size associated with each of methods 2-8 was adjusted until the error

$$E = \{[x(t_f) - x_R(t_f)]^2 + [y(t_f) - y_R(t_f)]^2 + [z(t_f) - z_R(t_f)]^2\}^{1/2} \tag{48}$$

was reduced to the ranges of either 1 m ± 10%, or 30 m ± 10%. The 1 meter runs can be considered as both a validation test and as a measure of how well the various formulations might function as high-precision integrators. The 30 meter runs provide a more realistic test, since this tolerance is consistent with the J_2 gravity model for the class of trajectories considered. Note that the error associated with method 9 is not controllable (except by carrying higher order terms); the average error for the 12 test trajectories was 23 m. The tests were performed on an IBM 370/158 computer in double-precision FORTRAN H extended with a compilation optimizer which minimizes execution time. A four-cycle Runge-Kutta routine was arbitrarily chosen for the integrations. Listed in Table 1 are the test results for one of the 12 trajectories, which is characterized by

Table 1 A comparison of nine motion prediction methods

Method No.	Execution times, s 1 m	30 m
1	0.02206	0.00977
2	0.01209	0.00638
3	0.01918	0.00874
4	0.01241	0.00645
5	0.01032	0.00541
6	0.02481	0.01198
7	0.01761	0.00993
8	0.03811	0.01920
9	–	0.00117[a]

[a] Actual error = 15.5 m.

$$[x(t_0), y(t_0), z(t_0), t_0] = \{2982.326920, 385.435210, 6045.109090, 0\}$$

and

$$[x(t_f), y(t_f), z(t_f), t_f] = \{-3518.470972, 3905.146116, 3811.149005, 1862.5428720\}$$

where all distances are in kilometers and times are in seconds.

Although the comparison of execution times depends upon the test case conditions, the integration routine, the error criterion, and the particular compiler utilized, several significant trends were observed:

1) The Euler parameter-asymptotic expansion method (method 9) is the clear winner in all 30 m error runs.

2) The KS-Cowell,[4] KS-Encke,[5] rectangular-Encke,[2] and rectangular Encke/variation of parameters[3] were the most efficient numerical integration methods in the present applications.

3) The Burdet-Cowell[6] and Euler parameter-Cowell[8] are clearly inferior methods.

4) The Burdet-Encke method[7] is marginal when compared to method 1, but its advantage increases as the flight time increases.

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